HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGY COBORDISMS OF A SURFACE(Complex Analysis and Geometry of Hyperbolic Spaces)

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HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGY COBORDISMS OF A SURFACE

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1. Introduction

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 0$ with one boundary component. A *homology cylinder* (*over* $\Sigma_{g,1}$) consists of a homology cobordism from $\Sigma_{g,1}$ to itself with markings of its boundary. We denote by $C_{g,1}$ the set of all diffeomorphism classes of homology cylinders. Stacking two homology cylinders gives a new one, and by this, we can endow $C_{g,1}$ with a monoid structure. A systematic study of $C_{g,1}$ was initiated by Habiro in [4], where $C_{g,1}$ appeared as a nice collection of 3-manifolds to which his clasper surgery theory is applied. Later Garoufalidis-Levine [3] and Levine [9] introduced a group $\mathcal{H}_{g,1}$ by taking a quotient of $C_{g,1}$ with respect to homology cobordant of homology cylinders. A feature of the monoid $C_{g,1}$ and the group $\mathcal{H}_{g,1}$ is that they contain the mapping class group $\mathcal{M}_{g,1}$, which is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,1}$. Moreover some tools for studying $\mathcal{M}_{g,1}$ can be also used for $C_{g,1}$ and $\mathcal{H}_{g,1}$ after appropriate generalizations. From these facts, we can consider $C_{g,1}$ and $\mathcal{H}_{g,1}$ to be enlargements of $\mathcal{M}_{g,1}$.

Now we consider an application of higher-order Alexander invariants, which are numerical invariants of finitely presentable groups, to homology cylinders. Higher-order Alexander invariants were first defined by Cochran in [1] for knot groups, and then generalized for arbitrary finitely presentable groups by Harvey in [5, 6]. They are interpreted as degrees of "non-commutative Alexander polynomials", which have some unclear ambiguity except their degrees in difficulties of non-commutative rings. Using them, Harvey obtained various sharper results than those given by the ordinary Alexander invariants — lower bounds on the Thurston norm, necessary conditions for realizing a given group as the fundamental group of some compact oriented 3-manifold, and so on.

In the process of applying higher-order Alexander invariants to homology cylinders, we can see that the Magnus representation for homology cylinders [15] plays an important role. This representation generalizes not only the Magnus representation for $\mathcal{M}_{g,1}$ defined by Morita [11], but the Gassner representation for string links given by Le Dimet [8] and Kirk-Livingston-Wang [7]. In this paper, we begin by reviewing the definition and fundamental properties of the Magnus representation, and then study some relationships to higher-order Alexander invariants. Note that the paper [16] treats the same topics and complements the contents of this paper.

2. Homology cobordisms of surfaces

We proceed all our discussion in PL or smooth category.

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 0$ with one boundary component. We take a base point p on the boundary of $\Sigma_{g,1}$, and take $2g$ loops $\gamma_1, \ldots, \gamma_{2g}$ of $\Sigma_{g,1}$ as shown in Figure 1. We consider them to be an embedded bouquet R_{2g} of $2g$-circles tied at the base point $p \in \partial\Sigma_{g,1}$. Then R_{2g} and the boundary loop ζ of $\Sigma_{g,1}$ together with one 2-cell make up a standard CW-decomposition of $\Sigma_{g,1}$. It is well-known that the fundamental group $\pi_1\Sigma_{g,1}$ of $\Sigma_{g,1}$ is isomorphic to the free group F_{2g} of rank $2g$ generated by $\gamma_1, \ldots, \gamma_{2g}$, in which $\zeta = \prod_{i=1}^{g}[\gamma_i, \gamma_{g+i}]$.

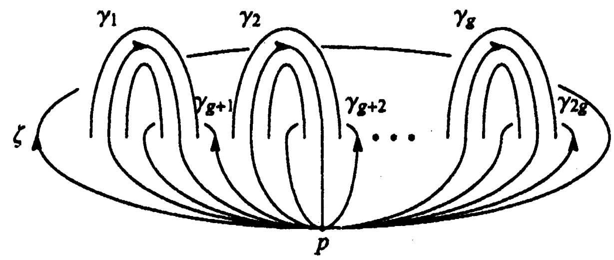

Figure 1

A *homology cylinder* (M, i_+, i_-) (*over* $\Sigma_{g,1}$), which has its origin in Habiro [4], Garoufalidis-Levine [3] and Levine [9], consists of a compact oriented 3-manifold M and two embeddings $i_+, i_- : \Sigma_{g,1} \to \partial M$ satisfying that

(1) i_+ is orientation-preserving and i_- is orientation-reversing,
(2) $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$,
(3) $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$,
(4) $i_+, i_- : H_*(\Sigma_{g,1}) \to H_*(M)$ are isomorphisms.

We denote $i_+(p) = i_-(p)$ by $p \in \partial M$ again and consider it to be the base point of M. We write a homology cylinder by (M, i_+, i_-) or simply by M.

Two homology cylinders are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism between the underlying 3-manifolds which is compatible with the markings. We denote the set of isomorphism classes of homology cylinders by $\mathcal{C}_{g,1}$. Given two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$, we can define a new homology cylinder $M \cdot N$ by

$$M \cdot N = (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$

Then $\mathcal{C}_{g,1}$ becomes a monoid with the identity element $1_{\mathcal{C}_{g,1}} := (\Sigma_{g,1} \times I, \mathrm{id} \times 1, \mathrm{id} \times 0)$.

From the monoid $\mathcal{C}_{g,1}$, we can construct the *homology cobordism group* $\mathcal{H}_{g,1}$ *of homology cylinders* as in the following way. Two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ are *homology cobordant* if there exists a compact oriented 4-manifold W such that

(1) $\partial W = M \cup (-N)/(i_+(x) = j_+(x),\ i_-(x) = j_-(x)) \quad x \in \Sigma_{g,1}$,
(2) the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ induce isomorphisms on the homology,

where $-N$ is N with opposite orientation. We denote by $\mathcal{H}_{g,1}$ the quotient set of $C_{g,1}$ with respect to the equivalence relation of homology cobordism. The monoid structure of $C_{g,1}$ induces a group structure of $\mathcal{H}_{g,1}$. In the group $\mathcal{H}_{g,1}$, the inverse of (M, i_+, i_-) is given by $(-M, i_-, i_+)$.

Example 2.1. For each element φ of the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$, we can construct a homology cylinder $M_\varphi \in C_{g,1}$ defined by

$$M_\varphi := (\Sigma_{g,1} \times I, \mathrm{id} \times 1, \varphi \times 0),$$

where collars of $i_+(\Sigma_{g,1})$ and $i_-(\Sigma_{g,1})$ are stretched half-way along $\partial\Sigma_{g,1} \times I$. This gives injective homomorphisms $\mathcal{M}_{g,1} \hookrightarrow C_{g,1}$ and $\mathcal{M}_{g,1} \hookrightarrow \mathcal{H}_{g,1}$.

Let $N_k(G) := G/(\Gamma^k G)$ be the k-th nilpotent quotient of a group G, where we define $\Gamma^1 G = G$ and $\Gamma^{i+1}G = [\Gamma^i G, G]$ for $i \geq 1$. For simplicity, we write $N_k(X)$ for $N_k(\pi_1 X)$ where X is a CW-complex, and write N_k for $N_k(F_{2g}) = N_k(\Sigma_{g,1})$. It is known that N_k is a torsion-free nilpotent group for each $k \geq 2$.

Let (M, i_+, i_-) be a homology cylinder. By definition, $i_+, i_- : \pi_1\Sigma_{g,1} \to \pi_1 M$ are both *2-connected*, namely they induce isomorphisms on H_1 and epimorphisms on H_2. Then, by Stallings' theorem [17], $i_+, i_- : N_k \xrightarrow{\cong} N_k(M)$ are isomorphisms for each $k \geq 2$. Using them, we obtain a monoid homomorphism

$$\sigma_k : C_{g,1} \longrightarrow \mathrm{Aut} N_k \qquad ((M, i_+, i_-) \mapsto (i_+)^{-1} \circ i_-).$$

It can be easily checked that σ_k induces a group homomorphism $\sigma_k : \mathcal{H}_{g,1} \to \mathrm{Aut}N_k$. We define filtrations of $C_{g,1}$ and $\mathcal{H}_{g,1}$ by

$$C_{g,1}[1] := C_{g,1}, \quad C_{g,1}[k] := \mathrm{Ker}\left(C_{g,1} \xrightarrow{\sigma_k} \mathrm{Aut}N_k\right) \text{ for } k \geq 2,$$
$$\mathcal{H}_{g,1}[1] := \mathcal{H}_{g,1}, \quad \mathcal{H}_{g,1}[k] := \mathrm{Ker}\left(\mathcal{H}_{g,1} \xrightarrow{\sigma_k} \mathrm{Aut}N_k\right) \text{ for } k \geq 2.$$

3. Magnus representations for homology cylinders

We first summarize our notation. For a matrix A with entries in a ring R, and a homomorphism $\varphi : R \to R'$, we denote by ${}^\varphi A$ the matrix obtained from A by applying φ to each entry. A^T denotes the transpose of A. When $R = \mathbf{Z}G$ for a group G or its right field of fractions (if exists), we denote by $\overline{A}$ the matrix obtained from A by applying the involution induced from $(x \mapsto x^{-1},\ x \in G)$ to each entry. For a module M, we write M^n (resp. M_n) for the module of column (resp. row) vectors with n entries.

For a finite CW-complex X and its regular covering X_Γ with respect to a homomorphism $\pi_1 X \to \Gamma$, Γ acts on X_Γ from the right through its deck transformation group. Therefore we regard the $\mathbf{Z}\Gamma$-cellular chain complex $C_*(X_\Gamma)$ of X_Γ as a collection of free right $\mathbf{Z}\Gamma$-modules consisting of column vectors together with differentials given by left multiplications of matrices. For each $\mathbf{Z}\Gamma$-bimodule A, the twisted chain complex $C_*(X; A)$ is given by the tensor product of the right $\mathbf{Z}\Gamma$-module $C_*(X_\Gamma)$ and the left $\mathbf{Z}\Gamma$-module A, so that $C_*(X; A)$ and $H_*(X; A)$ are right $\mathbf{Z}\Gamma$-modules.

Now we define and study the Magnus representation for homology cylinders. The following construction is based on Kirk-Livingston-Wang's paper [7].

Let $(M, i_+, i_-) \in C_{g,1}$ be a homology cylinder. By Stallings' theorem, N_k and $N_k(M)$ are isomorphic. Since N_k is a finitely generated torsion-free nilpotent group for each $k \geq 2$, we can embed $\mathbf{Z}N_k$ into the right field of fractions $\mathcal{K}_{N_k} := \mathbf{Z}N_k(\mathbf{Z}N_k - \{0\})^{-1}$. (See Section 5.) Similarly, we obtain $\mathbf{Z}N_k(M) \hookrightarrow \mathcal{K}_{N_k(M)} := \mathbf{Z}N_k(M)(\mathbf{Z}N_k(M) - \{0\})^{-1}$. We consider $\mathcal{K}_{N_k}$ (resp. $\mathcal{K}_{N_k(M)}$) to be a local coefficient system on $\Sigma_{g,1}$ (resp. M).

By a standard argument using covering spaces, we have the following.

Lemma 3.1. *$i_\pm : H_*(\Sigma_{g,1}, p; i_\pm^*\mathcal{K}_{N_k(M)}) \to H_*(M, p; \mathcal{K}_{N_k(M)})$ are isomorphisms as right $\mathcal{K}_{N_k(M)}$-vector spaces.*

Since $R_{2g} \subset \Sigma_{g,1}$ is a deformation retract, we have

$$H_1(\Sigma_{g,1}, p; i_\pm^*\mathcal{K}_{N_k(M)}) \cong H_1(R_{2g}, p; i_\pm^*\mathcal{K}_{N_k(M)}) = C_1(\widetilde{R_{2g}}) \otimes_{F_{2g}} i_\pm^*\mathcal{K}_{N_k(M)} \cong \mathcal{K}_{N_k(M)}^{2g}$$

with a basis

$$\{\overline{\gamma_1} \otimes 1, \ldots, \overline{\gamma_{2g}} \otimes 1\} \subset C_1(\widetilde{R_{2g}}) \otimes_{F_{2g}} i_\pm^*\mathcal{K}_{N_k(M)}$$

as a right $\mathcal{K}_{N_k(M)}$-vector space, where $\overline{\gamma_i}$ is a lift of γ_i on the universal covering $\widetilde{R_{2g}}$.

Definition 3.2. (1) For each $M = (M, i_+, i_-) \in C_{g,1}$, we denote by $r_k'(M) \in GL(2g, \mathcal{K}_{N_k(M)})$ the representation matrix of the right $\mathcal{K}_{N_k(M)}$-isomorphism

$$\mathcal{K}_{N_k(M)}^{2g} \cong H_1(\Sigma_{g,1}, p; i_-^*\mathcal{K}_{N_k(M)}) \xrightarrow[i_-]{\cong} H_1(M, p; \mathcal{K}_{N_k(M)}) \xrightarrow[i_+^{-1}]{\cong} H_1(\Sigma_{g,1}, p; i_+^*\mathcal{K}_{N_k(M)}) \cong \mathcal{K}_{N_k(M)}^{2g}$$

(2) The *Magnus representation* for $C_{g,1}$ is the map $r_k : C_{g,1} \to GL(2g, \mathcal{K}_{N_k})$ which assigns to $M = (M, i_+, i_-) \in C_{g,1}$ the matrix ${}^{i_+^{-1}}r_k'(M)$.

While we call $r_k(M)$ the Magnus "representation", it is actually a crossed homomorphism.

Theorem 3.3 ([14, Theorem 7.12]). *For $M_1 = (M_1, i_+, i_-)$, $M_2 = (M_2, j_+, j_-) \in C_{g,1}$, we have*

$$r_k(M_1 \cdot M_2) = r_k(M_1) \cdot {}^{\sigma_k(M_1)}r_k(M_2).$$

Moreover, we can show the following.

Theorem 3.4 ([14, Theorem 7.13]). *$r_k : C_{g,1} \to GL(2g, \mathcal{K}_{N_k})$ factors through $\mathcal{H}_{g,1}$.*

Consequently, we obtain the Magnus representation $r_k : \mathcal{H}_{g,1} \to GL(2g, \mathcal{K}_{N_k})$, which is a crossed homomorphism. Note that if we restrict r_k to $C_{g,1}[k]$ (and $\mathcal{H}_{g,1}[k]$), it becomes a homomorphism.

Example 3.5. For $\varphi \in \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}F_{2g}$, we can obtain

$$r_k(M_\varphi) = \overline{{}^{\rho_k}\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}\right)_{i,j}},$$

where $\rho_k : \mathbf{Z}F_{2g} \to \mathbf{Z}N_k \subset \mathcal{K}_{N_k}$ is the natural map and $\partial/\partial\gamma_i$ are free differentials. From this, we see that r_k generalizes the original Magnus representation for $\mathcal{M}_{g,1}$ in [11].

In general, the Magnus matrix $r_k(M)$ of a homology cylinder M can be obtained from a finite presentation of the form

$$\pi_1 M \cong \left\langle \begin{array}{l} i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), \\ z_1, \ldots, z_{2g+l}, \\ i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) \end{array} \middle| \begin{array}{l} i_-(\gamma_1)s_1, \ldots, i_-(\gamma_{2g})s_{2g}, \\ r_1, \ldots r_l, \\ i_+(\gamma_1)u_1, \ldots, i_+(\gamma_{2g})u_{2g} \end{array} \right\rangle,$$

where s_i, r_i and u_i are words in $z_1, \dots, z_{2g+l}$, by a purely algebraic calculation. Note that such a presentation does exist for each homology cylinder.

As in the case of $\mathcal{M}_{g,1}$ (see [11] and [18]), the Magnus representation for $\mathcal{H}_{g,1}$ satisfies the following "symplectic" property.

Theorem 3.6. *For any homology cylinder M, we have the equality*

$$\overline{r_k(M)^T}\,\widetilde{J}\,r_k(M) = {}^{\sigma_k(M)}\widetilde{J},$$

where $\widetilde{J} = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} \in GL(2g, \mathbb{Z}N_k)$ *is defined by*

$$J_1 = \begin{pmatrix} 1-\gamma_1 & & & & 0 \\ (1-\gamma_2)(1-\gamma_1^{-1}) & 1-\gamma_2 & & & \\ (1-\gamma_3)(1-\gamma_1^{-1}) & (1-\gamma_3)(1-\gamma_2^{-1}) & 1-\gamma_3 & & \\ \vdots & \vdots & & \ddots & \\ (1-\gamma_g)(1-\gamma_1^{-1}) & (1-\gamma_g)(1-\gamma_2^{-1}) & \cdots & & 1-\gamma_g \end{pmatrix},$$

$$J_2 = \begin{pmatrix} \gamma_1\gamma_{g+1}^{-1} & & & & 0 \\ (1-\gamma_2)(1-\gamma_{g+1}^{-1}) & \gamma_2\gamma_{g+2}^{-1} & & & \\ (1-\gamma_3)(1-\gamma_{g+1}^{-1}) & (1-\gamma_3)(1-\gamma_{g+2}^{-1}) & \gamma_3\gamma_{g+3}^{-1} & & \\ \vdots & \vdots & & \ddots & \\ (1-\gamma_g)(1-\gamma_{g+1}^{-1}) & (1-\gamma_g)(1-\gamma_{g+2}^{-1}) & \cdots & & \gamma_g\gamma_{2g}^{-1} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 1-\gamma_1^{-1}-\gamma_{g+1} & & & & 0 \\ (1-\gamma_{g+2})(1-\gamma_1^{-1}) & 1-\gamma_2^{-1}-\gamma_{g+2} & & & \\ (1-\gamma_{g+3})(1-\gamma_1^{-1}) & (1-\gamma_{g+3})(1-\gamma_2^{-1}) & 1-\gamma_3^{-1}-\gamma_{g+3} & & \\ \vdots & \vdots & & \ddots & \\ (1-\gamma_{2g})(1-\gamma_1^{-1}) & (1-\gamma_{2g})(1-\gamma_2^{-1}) & \cdots & & 1-\gamma_g^{-1}-\gamma_{2g} \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 1-\gamma_{g+1}^{-1} & & & & 0 \\ (1-\gamma_{g+2})(1-\gamma_{g+1}^{-1}) & 1-\gamma_{g+2}^{-1} & & & \\ (1-\gamma_{g+3})(1-\gamma_{g+1}^{-1}) & (1-\gamma_{g+3})(1-\gamma_{g+2}^{-1}) & 1-\gamma_{g+3}^{-1} & & \\ \vdots & \vdots & & \ddots & \\ (1-\gamma_{2g})(1-\gamma_{g+1}^{-1}) & (1-\gamma_{2g})(1-\gamma_{g+2}^{-1}) & \cdots & & 1-\gamma_{2g}^{-1} \end{pmatrix}.$$

Note that the matrix $\widetilde{J}$ appeared in Papakyriakopoulos' paper [12], and that it is mapped to the ordinary symplectic matrix by the augmentation map $\mathbb{Z}N_k \to \mathbb{Z}$.

Sketch of Proof. First we define a natural pairing

$$\langle\, ,\, \rangle : H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) \times H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k}) \longrightarrow \mathcal{K}_{N_k}$$

satisfying

$$\langle af, b\rangle = \overline{f}\langle a, b\rangle, \quad \langle a, bf\rangle = \langle a, b\rangle f$$

for all $f \in \mathcal{K}_{N_k}$. This generalizes Suzuki's higher intersection form in [18]. To construct it, we use the following type of the Poincaré-Lefschetz duality: *Let X be a compact oriented n-manifold whose boundary ∂M is decomposed as the union of two compact manifolds A and B*

with $\partial A = \partial B = A \cap B$, and let M be a local coefficient system on X. Then the cap product with a fundamental class gives isomorphisms $H^k(X, A; M) \xrightarrow{\cong} H_{n-k}(X, B; M)$ for all k.

The naturality of the Poincaré-Lefschetz duality shows the equality

$$\langle r_k(M)a, r_k(M)b \rangle = {}^{\sigma_k(M)}\langle a, b \rangle$$

for each homology cylinder M. By writing down this equality with respect to the basis $\{\overline{\gamma_1} \otimes 1, \ldots, \overline{\gamma_{2g}} \otimes 1\}$ of $H_1(\Sigma_{g,1}, p; \mathcal{K}_{N_k})$, where we use Papakyrirakopoulos' argument in [12], we obtain the desired equality. □

4. Example: Relationship to the Gassner representation for string links

In [9], Levine gave a method for constructing homology cylinders from pure string links. By this, we can obtain many homology cylinders not belonging to the subgroup $\mathcal{M}_{g,1}$. Also, we can see a relationship between the Gassner representation for string links and our representation.

For a g-component pure string link $L \subset D^2 \times I$, we now construct a homology cylinder $M_L \in \mathcal{C}_{g,1}$ as follows. Consider a closed tubular neighborhood of the loops $\gamma_{g+1}, \gamma_{g+2}, \ldots, \gamma_{2g}$ in Figure 1 to be the image of an embedding $\iota : D_g \hookrightarrow \Sigma_{g,1}$ of a g-holed disk D_g as in Figure 2.

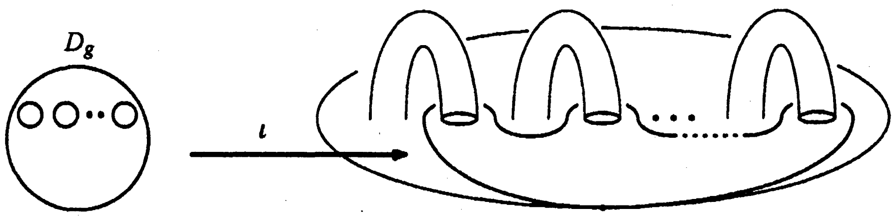

Figure 2

Let C be the complement of an open tubular neighborhood of L in $D^2 \times I$. For each choice a framing of L, a homeomorphism $h : \partial C \xrightarrow{\cong} \partial(\iota(D_g) \times I)$ is fixed. Then the manifold M_L given from $\Sigma_{g,1} \times I$ by removing $\iota(D_g) \times I$ and regluing C by h becomes a homology cylinder. This construction gives an injective monoid homomorphism $\mathcal{L}_g \to \mathcal{C}_{g,1}$ from the monoid $\mathcal{L}_g$ of (framed) pure string links to $\mathcal{C}_{g,1}$. Moreover it also induces an injective homomorphism $\mathcal{S}_g \to \mathcal{H}_{g,1}$ from the concordance group of (framed) pure string links to $\mathcal{H}_{g,1}$. In particular, the (smooth) knot concordance group, which coincides with $\mathcal{S}_1$, is embedded in $\mathcal{H}_{g,1}$. If we restrict these embeddings to the pure braid group, which is a subgroup of $\mathcal{L}_g$ and $\mathcal{S}_g$, their images are contained in $\mathcal{M}_{g,1}$.

We fix an integer $k \geq 2$. By the Gassner representation, we mean the crossed homomorphism $r_{G,k} : \mathcal{L}_g \to GL(g, \mathcal{K}_{N_k(D_g)})$ or $r_{G,k} : \mathcal{S}_g \to GL(g, \mathcal{K}_{N_k(D_g)})$ given by a construction similar to that in the previous section. (In [8] and [7], only $r_{G,2}$ is treated.) Comparing methods for calculating the Gassner and the Magnus representations, we obtain the following.

Theorem 4.1 ([14, Theorem 7.18]). *For any pure string link $L \in \mathcal{L}_g$, $r_k(M_L) = \begin{pmatrix} * & 0_g \\ * & r_{G,k}(L) \end{pmatrix}$.*

We mention two remarks about this theorem. First we identify $F_g = \pi_1 D_g$ with the subgroup of $F_{2g} = \pi_1 \Sigma_{g,1}$ generated by $\gamma_{g+1}, \dots, \gamma_{2g}$. Then the maps $F_g = \langle \gamma_{g+1}, \dots, \gamma_{2g} \rangle \hookrightarrow F_{2g} \twoheadrightarrow F_g$, where the second map sends $\gamma_1, \dots, \gamma_g$ to 1, show that $N_k(F_g) \subset N_k$ and $\mathcal{K}_{N_k(F_g)} \subset \mathcal{K}_{N_k}$. Second, the embeddings $\mathcal{L}_g \hookrightarrow C_{g,1}$ and $\mathcal{S}_g \hookrightarrow \mathcal{H}_{g,1}$ have ambiguity with respect to framings. However we can check that the lower right part of $r_k(M_L)$ does not depend on the choice of framings.

Corollary 4.2. *$\mathcal{M}_{g,1}$ is not a normal subgroup of $\mathcal{H}_{g,1}$ for $g \geq 3$.*

Proof. In [7], they gave 3-component pure string links denoted by L_5 and L_6 having the condition that L_5 is a pure braid, while the conjugate $L_6 L_5 L_6^{-1}$ is not. To show that $L_6 L_5 L_6^{-1}$ is not a pure braid, they use the fact that $r_{G,2}(L_6 L_5 L_6^{-1})$ has an entry not belonging to $\mathbb{Z}N_2(D_3)$. Then our claim follows from Theorem 4.1 with respect to this example. □

Example 4.3. Let L be a 2-component pure string link as depicted in Figure 3.

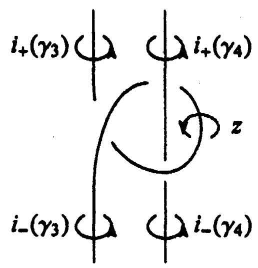

Figure 3

Then the presentation of $\pi_1 M_L$ is given by

$$\pi_1 M_L \cong \left\langle \begin{array}{c} i_-(\gamma_1), \dots, i_-(\gamma_4) \\ z \\ i_+(\gamma_1), \dots, i_+(\gamma_4) \end{array} \middle| \begin{array}{l} i_+(\gamma_1) i_-(\gamma_3)^{-1} i_+(\gamma_4) i_-(\gamma_1)^{-1}, \\ [i_+(\gamma_1), i_+(\gamma_3)] i_+(\gamma_2) z i_-(\gamma_2)^{-1} [i_-(\gamma_3), i_-(\gamma_1)], \\ i_+(\gamma_4) i_-(\gamma_3) i_+(\gamma_4)^{-1} z^{-1}, \quad i_-(\gamma_3) i_+(\gamma_3)^{-1} i_-(\gamma_3)^{-1} z, \\ i_-(\gamma_4) z^{-1} i_+(\gamma_4)^{-1} z, \end{array} \right\rangle,$$

where we use the blackboard framing. We identify N_2 and $N_2(M_L)$ by using i_+. Using the presentation, we have $z = i_-(\gamma_3) = \gamma_3$, $i_-(\gamma_4) = \gamma_4$, $i_-(\gamma_2) = \gamma_2\gamma_3$ and $i_-(\gamma_1) = \gamma_1 \gamma_3^{-1} \gamma_4$ in N_2. Then we obtain

$$r_2(M_L) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-\gamma_1^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}-\gamma_4^{-1}+1}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_3^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_4^{-1}(\gamma_4^{-1}-1)}{\gamma_3^{-1}+\gamma_4^{-1}-1} \\ \frac{\gamma_1^{-1}\gamma_3\gamma_4^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{(1-\gamma_3^{-1})(\gamma_2^{-1}\gamma_3^{-1}-\gamma_2^{-1}-1)}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_3^{-1}-1}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{-\gamma_3^{-1}\gamma_4^{-1}+\gamma_3^{-1}+2\gamma_4^{-1}-1}{\gamma_3^{-1}+\gamma_4^{-1}-1} \end{pmatrix}.$$

Note that $\det r_2(M_L) = \dfrac{\gamma_3 + \gamma_4 - 1}{\gamma_3\gamma_4(\gamma_3^{-1} + \gamma_4^{-1} - 1)}$.

5. Higher-order Alexander invariants and torsion-degree functions

Here we summarize the theory of higher-order Alexander invariants along the lines of Harvey's papers [5, 6]. For our use, we generalize them to functions of matrices called *torsion-degree functions*.

A group Γ is *poly-torsion-free-abelian* (PTFA, for short) if Γ has a normal series of finite length whose successive quotients are all torsion-free abelian. In particular, free nilpotent quotients N_k are PTFA for all $k \geq 2$. Note that any subgroup of a PTFA group is also PTFA.

For each PTFA group Γ, the group ring $\mathbf{Z}\Gamma$ is known to be an Ore domain, so that it can be embedded in the *right field of fractions* $\mathcal{K}_\Gamma := \mathbf{Z}\Gamma(\mathbf{Z}\Gamma - \{0\})^{-1}$, which is a skew field. We refer to [2], [13] for localizations of non-commutative rings.

We will also use the following localizations of $\mathbf{Z}\Gamma$ placed between $\mathbf{Z}\Gamma$ and $\mathcal{K}_\Gamma$. Let $\psi \in H^1(\Gamma)$ be a primitive element. This means the corresponding homomorphism, which is denoted by ψ again, under $H^1(\Gamma) \cong \mathrm{Hom}(\Gamma, \mathbf{Z})$ is onto. Then we have an exact sequence

$$1 \longrightarrow (\Gamma^\psi := \mathrm{Ker}\,\psi) \longrightarrow \Gamma \xrightarrow{\psi} \mathbf{Z} \longrightarrow 1.$$

We take a splitting $\xi : \mathbf{Z} \to \Gamma$ of this sequence and put $t := \xi(1) \in \Gamma$. Since Γ^ψ is again a PTFA group, $\mathbf{Z}\Gamma^\psi$ can be embedded in its right field of fractions $\mathcal{K}_{\Gamma^\psi} = \mathbf{Z}\Gamma^\psi(\mathbf{Z}\Gamma^\psi - \{0\})^{-1}$. Moreover, we can construct a right quotient ring $\mathbf{Z}\Gamma(\mathbf{Z}\Gamma^\psi - \{0\})^{-1}$. Then the splitting ξ gives an isomorphism between $\mathbf{Z}\Gamma(\mathbf{Z}\Gamma^\psi - \{0\})^{-1}$ and the skew Laurent polynomial ring $\mathcal{K}_{\Gamma^\psi}[t^\pm]$, in which $at = t(t^{-1}at)$ holds for each $a \in \Gamma$. $\mathcal{K}_{\Gamma^\psi}[t^\pm]$ is known to be a non-commutative right and left principal ideal domain. By definition, we have inclusions

$$\mathbf{Z}\Gamma \hookrightarrow \mathcal{K}_{\Gamma^\psi}[t^\pm] \hookrightarrow \mathcal{K}_\Gamma.$$

$\mathcal{K}_{\Gamma^\psi}[t^\pm]$ and $\mathcal{K}_\Gamma$ are known to be flat $\mathbf{Z}\Gamma$-modules. On $\mathcal{K}_{\Gamma^\psi}[t^\pm]$, we have a map $\deg^\psi : \mathcal{K}_{\Gamma^\psi}[t^\pm] \to \mathbf{Z}_{\geq 0} \cup \{\infty\}$ assigning to each polynomial its degree. We put $\deg^\psi(0) := \infty$. Note that the composite $\mathbf{Z}\Gamma(\mathbf{Z}\Gamma^\psi - \{0\})^{-1} \xrightarrow{\cong} \mathcal{K}_{\Gamma^\psi}[t^\pm] \xrightarrow{\deg^\psi} \mathbf{Z}_{\geq 0} \cup \{\infty\}$ does not depend on the choice of the splitting ξ.

Harvey's higher-order Alexander invariants [6] are defined as follows. Let G be a finitely presentable group, and let $\varphi : G \twoheadrightarrow \mathbf{Z}$ be an epimorphism. For a PTFA group Γ and an epimorphism $\varphi_\Gamma : G \twoheadrightarrow \Gamma$, $(\varphi_\Gamma, \varphi)$ is called an *admissible pair* for G if there exists an epimorphism $\psi : \Gamma \twoheadrightarrow \mathbf{Z}$ satisfying $\varphi = \psi \circ \varphi_\Gamma$. For each admissible pair $(\varphi_\Gamma, \varphi)$ for G, we regard $\mathcal{K}_{\Gamma^\psi}[t^\pm] = \mathbf{Z}\Gamma(\mathbf{Z}\Gamma^\psi - \{0\})^{-1}$ as a $\mathbf{Z}G$-module, and we define the higher-order Alexander invariant for $(\varphi_\Gamma, \varphi)$ by

$$\overrightarrow{\delta}_\Gamma^\psi(G) = \dim_{\mathcal{K}_{\Gamma^\psi}}(H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm])) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$$

$\overrightarrow{\delta}_\Gamma^\psi(G)$ is also called the Γ-*degree*[1]. Note that the right $\mathcal{K}_{\Gamma^\psi}[t^\pm]$-module $H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm])$ are decomposed into

$$H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm]) = (\mathcal{K}_{\Gamma^\psi}[t^\pm])^r \oplus \left(\bigoplus_{i=1}^{l} \frac{\mathcal{K}_{\Gamma^\psi}[t^\pm]}{p_i(t)\mathcal{K}_{\Gamma^\psi}[t^\pm]} \right)$$

[1] Our definition is slightly different from that in [6].

for some $r \in \mathbf{Z}_{\geq 0}$ and $p_i(t) \in \mathcal{K}_{\Gamma_\psi}[t^{\pm}]$, and then

$$\overrightarrow{\delta}_\Gamma^\psi(G) = \begin{cases} \sum_{i=1}^{l} \deg^\psi(p_i(t)) & (r = 0), \\ \infty & (r > 0) \end{cases}$$

For a space X and an admissible pair for $\pi_1 X$, we define $\overrightarrow{\delta}_\Gamma^\psi(X) := \overrightarrow{\delta}_\Gamma^\psi(\pi_1 X)$.

For a finitely presentable group G and an admissible pair $(\varphi_\Gamma, \varphi)$ for G. The Γ-degree can be computed from any presentation matrix of the right $\mathcal{K}_{\Gamma_\psi}[t^{\pm}]$-module $H_1(G; \mathcal{K}_{\Gamma_\psi}[t^{\pm}])$. Therefore we can consider it to be a $\mathbf{Z}_{\geq 0}$-valued function on the set $M(\mathcal{K}_{\Gamma_\psi}[t^{\pm}])$ of all matrices with entries in $\mathcal{K}_{\Gamma_\psi}[t^{\pm}]$. In [14] (see also [16]), we extended this function to

$$\overrightarrow{d}_\Gamma^\psi : M(\mathcal{K}_\Gamma) \to \mathbf{Z} \cup \{\infty\}$$

called the *(truncated) torsion-degree function* by using Reidemeister torsions and the Dieudonné determinant $\det : GL(\mathcal{K}_\Gamma) \to (\mathcal{K}_\Gamma^\times)_{\mathrm{ab}}$, where $(\mathcal{K}_\Gamma^\times)_{\mathrm{ab}}$ is the abelianization of the multiplicative group $\mathcal{K}_\Gamma^\times = \mathcal{K}_\Gamma - \{0\}$. The torsion-degree function is defined for each pair of a PTFA group Γ and an epimorphism $\psi : \Gamma \twoheadrightarrow \mathbf{Z}$. It can be regarded as a generalization of the extension of $\deg^\psi : \mathcal{K}_{\Gamma_\psi}[t^{\pm}] \to \mathbf{Z}_{\geq 0} \cup \{\infty\}$ to $\deg^\psi : \mathcal{K}_\Gamma \to \mathbf{Z} \cup \{\infty\}$ by setting $\deg^\psi(fg^{-1}) = \deg^\psi(f) - \deg^\psi(g)$ for $f \in \mathbf{Z}\Gamma, g \in \mathbf{Z}\Gamma - \{0\}$ (see Proposition 9.1.1 in [2], for example). It induces a group homomorphism $\deg^\psi : (\mathcal{K}_\Gamma^\times)_{\mathrm{ab}} \to \mathbf{Z}$.

Torsion-degree functions have the following properties.

Proposition 5.1. (1) *For $A \in GL(\mathcal{K}_\Gamma)$, we have $\overrightarrow{d}_\Gamma^\psi(A) = \deg^\psi(\det A)$. In particular, $\overrightarrow{d}_\Gamma^\psi(A) = 0$ for any $A \in GL(\mathcal{K}_{\Gamma_\psi}[t^{\pm}])$.*
(2) *Let M be a finitely generated right $\mathcal{K}_{\Gamma_\psi}[t^{\pm}]$-module presented by a matrix $A \in M(\mathcal{K}_{\Gamma_\psi}[t^{\pm}])$. Then*

$$\overrightarrow{d}_\Gamma^\psi(A) = \begin{cases} \dim_{\mathcal{K}_{\Gamma_\psi}}(T_{\mathcal{K}_{\Gamma_\psi}[t^{\pm}]} M) & (\mathrm{rank}_{\mathcal{K}_{\Gamma_\psi}[t^{\pm}]}(F_{\mathcal{K}_{\Gamma_\psi}[t^{\pm}]} M) \leq 1) \\ \infty & (\text{otherwise}) \end{cases},$$

where $T_{\mathcal{K}_{\Gamma_\psi}[t^{\pm}]} M$ (resp. $F_{\mathcal{K}_{\Gamma_\psi}[t^{\pm}]} M$) is the $\mathcal{K}_{\Gamma_\psi}[t^{\pm}]$-torsion (resp. $\mathcal{K}_{\Gamma_\psi}[t^{\pm}]$-free) part of M.

Let G be a finitely presentable group and we take a presentation $\langle x_1, \ldots, x_l \mid r_1, \ldots, r_m \rangle$ of G. For each admissible pair $(\varphi_\Gamma, \varphi)$ for G, the Jacobi matrix $A := {}^{\varphi_\Gamma}\!\left(\dfrac{\partial r_j}{\partial x_i}\right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}}$ at $\mathcal{K}_{\Gamma_\psi}[t^{\pm}]$ gives a presentation matrix of $H_1(G, \{1\}; \mathcal{K}_{\Gamma_\psi}[t^{\pm}])$. Then the Γ-degree is given by

$$\overrightarrow{\delta}_\Gamma^\psi(G) = \dim_{\mathcal{K}_{\Gamma_\psi}}(H_1(G; \mathcal{K}_{\Gamma_\psi}[t^{\pm}])) = \overrightarrow{d}_\Gamma^\psi(A),$$

where the second equality follows from the direct sum decomposition

$$H_1(G, \{1\}; \mathcal{K}_{\Gamma_\psi}[t^{\pm}]) \cong H_1(G; \mathcal{K}_{\Gamma_\psi}[t^{\pm}]) \oplus \mathcal{K}_{\Gamma_\psi}[t^{\pm}]$$

given by Harvey in [5].

6. Applications of torsion-degree functions to homology cylinders

In this section, we study some invariants of homology cylinders arising from the Magnus representation, twisted homology groups of related manifolds and torsion-degree functions. In [14], we can see other applications.

6.1. **Torsion-degrees of Magnus matrices.** First, we consider torsion-degree functions associated to nilpotent quotients N_k of $\pi_1\Sigma_{g,1}$, and apply them to Magnus matrices. Since $H_1(N_k) = H_1(N_2) = H_1(\Sigma_{g,1})$ and $H^1(N_k) = H^1(N_2) = H^1(\Sigma_{g,1})$, taking an epimorphism $N_k \twoheadrightarrow \mathbf{Z}$, which is needed in the definition of a torsion-degree function, is done by choosing a primitive element of $H^1(\Sigma_{g,1})$.

Theorem 6.1. *Let M be a homology cylinder. For any $k \geq 2$ and any primitive element $\psi \in H^1(\Sigma_{g,1})$, the torsion-degree $\overrightarrow{d}^{\psi}_{N_k}(r_k(M))$ is always zero.*

Proof. Proposition 5.1 (1) shows that torsion-degrees are additive for products of invertible matrices and vanish for those in $GL(\mathbf{Z}N_k)$. It can be also checked that they are invariant under taking the transpose and operating the involution. Hence, by applying the torsion-degree function to the equality $\overline{r_k(M)^T}\,\overline{J} r_k(M) = {}^{\sigma_k(M)}\overline{J}$ in Theorem 3.6, we obtain $2\overrightarrow{d}^{\psi}_{N_k}(r_k(M)) = 0$. This completes the proof. □

Example 6.2. Consider the homology cylinder M_L in Example 4.3. $\overrightarrow{d}^{\psi}_{N_2}(r_2(M_L))$ is given by the degree of $\det r_2(M_L) = \frac{\gamma_3+\gamma_4-1}{\gamma_3\gamma_4(\gamma_3^{-1}+\gamma_4^{-1}-1)}$ with respect to ψ. It can be easily checked that it is zero.

Remark 6.3. In [14], we defined the Magnus representation $r_k : \mathrm{Aut}F_n^{\mathrm{acy}} \to GL(n, \mathcal{K}_{N_k(F_n)})$ for $\mathrm{Aut}F_n^{\mathrm{acy}}$, where F_n^{acy} is a completion of F_n in a certain sense and is called the *acyclic closure* of F_n. The natural map $F_n \to F_n^{\mathrm{acy}}$ is known to be injective and 2-connected. In particular, $N_k(F_n) = N_k(F_n^{\mathrm{acy}})$. $\mathrm{Aut}F_n^{\mathrm{acy}}$ can be regarded as an enlargement of $\mathrm{Aut}F_n$, and we have the enlarged Dehn-Nielsen homomorphism $\sigma^{\mathrm{acy}} : \mathcal{H}_{g,1} \to \mathrm{Aut}F_{2g}^{\mathrm{acy}}$ extending the classical one $\sigma : \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}F_{2g}$. (Note that σ^{acy} is not injective.) That is, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Aut}F_{2g} & \hookrightarrow & \mathrm{Aut}F_{2g}^{\mathrm{acy}} \\ \uparrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle\sigma^{\mathrm{acy}}} \\ \mathcal{M}_{g,1} & \hookrightarrow & \mathcal{H}_{g,1} \end{array}$$

The Magnus representation for homology cylinders is nothing other than the composite $\mathcal{H}_{g,1} \xrightarrow{\sigma^{\mathrm{acy}}} \mathrm{Aut}F_{2g}^{\mathrm{acy}} \xrightarrow{r_k} GL(2g, \mathcal{K}_{N_k})$. We can easily check that $\overrightarrow{d}^{\psi}_{N_k} \circ r_k : \mathrm{Aut}F_{2g}^{\mathrm{acy}} \xrightarrow{r_k} GL(2g, \mathcal{K}_{N_k})$ is nontrivial. Therefore $\overrightarrow{d}^{\psi}_{N_k} \circ r_k$ gives an invariant of $\mathrm{Aut}F_n^{\mathrm{acy}}$ which vanishes on $\mathcal{M}_{g,1}$, $\mathrm{Aut}F_n$ and $\mathcal{H}_{g,1}$ for each $k \geq 2$ and each primitive element $\psi \in H^1(N_k)$.

6.2. **Factorization formula of $N_{k,T}$-degree for the mapping torus of a homology cylinder.** For each homology cylinder $M = (M, i_+, i_-)$, we can construct a closed 3-manifold T_M as follows. First we attach a 2-handle $I \times D^2$ along $I \times i_{\pm}(\partial\Sigma_{g,1})$, so that we obtain a homology cylinder (M', i'_+, i'_-) over a closed surface Σ_g, which corresponds to the embedding $\Sigma_{g,1} \hookrightarrow \Sigma_g$. Then we put

$$T_M := M'/(i'_+(x) = i'_-(x)), \quad x \in \Sigma_g.$$

We call T_M the *mapping torus* of M. Indeed, for $M_\varphi \in \mathcal{M}_{g,1} \subset \mathcal{C}_{g,1}$, the resulting manifold T_{M_φ} is nothing other than the usual mapping torus of φ extended naturally to the mapping class of Σ_g. If $M \in \mathcal{C}_{g,1}[k]$, we have natural isomorphisms $N_k(\Sigma_g) \cong N_k(M')$ and $N_k(T_M) \cong N_k(\Sigma_g) \times \langle\lambda\rangle$.

Note that these groups are torsion-free nilpotent (hence PTFA). We consider $N_k(\Sigma_g)$ to be a subgroup of $N_k(T_M)$. For simplicity, we denote $N_k(T_M)$ by $N_{k,T}$.

By an argument similar to that in Lemma 3.1, we can show that $H_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_{k,T}}) = 0$. Hence we can define the Reidemeister torsion

$$\tau_{N_{k,T}}(M) := \tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_{k,T}})) \in K_1(\mathcal{K}_{N_{k,T}})/(\pm N_{k,T}).$$

(See [10], [19] for generalities of Reidemeister torsions) Then we obtain the following factorization formula of $N_{k,T}$-degree for the mapping torus of a homology cylinder.

Theorem 6.4 ([14, Theorem 11.6]). *Let M be a homology cylinder belonging to $C_{g,1}[k]$.*
(1) *For each primitive element $\psi \in H^1(N_{k,T}) = H^1(T_M)$, the $N_{k,T}$-degree $\overline{\delta}^{\psi}_{N_{k,T}}(T_M)$ is finite.*
(2) *We have the equality*

$$\overline{\delta}^{\psi}_{N_{k,T}}(T_M) = \overline{d}^{\psi}_{N_{k,T}}(\tau_{N_{k,T}}(M)) + \overline{d}^{\psi}_{N_{k,T}}\left(\lambda I_{2g} - \overline{r_{k,T}(M)^T}\right) - 2|\psi(\lambda)|,$$

where $r_{k,T} : \mathcal{H}_{g,1} \to GL(2g, \mathcal{K}_{N_{k,T}})$ is defined similarly to the Magnus representation r_k.

Remark 6.5 (The case of $k = 2$). Since $\mathbf{Z}N_{2,T} = \mathbf{Z}N_2(T_M)$ and $\mathcal{K}_{N_{2,T}} = \mathcal{K}_{N_2(T_M)}$ are commutative, we can use the ordinary determinant to calculate the invariants seen above. For $M \in C_{g,1}[2]$, we write Δ_{T_M} for the Alexander polynomial of T_M. By a straightforward computation, we have

$$\Delta_{T_M} \doteq \overline{\tau_{N_{2,T}}(M)} \cdot \det\left(\lambda I_{2g} - \overline{r_{2,T}(M)^T}\right) \cdot (1-\lambda)^{-2},$$

where $\doteq$ means that these equalities hold in $\mathcal{K}_{N_2(T_M)}$ up to $\pm N_2(T_M)$.

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